University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 12
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1 *Purcell 9.3   A free proton was at rest at the origin before the wave*

$$\mathbf{E} = \frac{(5\ statvolt/cm)\,\hat{\mathbf{y}}}{1 + [k(x + ct)]^2} \qquad \mathbf{B} = \frac{(-5\ gauss)\,\hat{\mathbf{z}}}{1 + [k(x + ct)]^2}$$

came past with $k = 1\ cm^{-1}$. Where would you expect to find the proton after 1 µs? The proton mass is $1.6 \times 10^{-24}\ g$.

To begin, we will neglect the magnetic force and see later if this is justified. In this case, the impulse due to the electric force will be in the y direction. The pulse only has an appreciable magnitude for a few nanoseconds, so we may extend the integral to infinity.

$$\Delta\mathbf{p} = \int \mathbf{F}_e dt = e\,(5\ statvolt/cm)\hat{\mathbf{y}} \int_{-\infty}^{\infty} \frac{dt}{1 + (kct)^2} = \frac{(5\ statvolt/cm)\pi e}{kc}\,\hat{\mathbf{y}}$$

$$\Delta\mathbf{p} = (2.5 \times 10^{-19}\ g\,cm/s)\,\hat{\mathbf{y}}$$

This corresponds to a speed of

$$v = 1.6 \times 10^5\ cm/s\ .$$

From the Lorentz force law

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}\ ,$$

we see that because the electric and magnetic fields have the same strength, the magnetic force is smaller by

$$F_B \simeq \frac{v}{c}F_e \simeq (5 \times 10^{-4})F_e\ ,$$

so our approximation is pretty good. The acceleration while the pulse is passing occurs for a very small time, so the position of the proton after one microsecond is essentially

$$y = (1.6 \times 10^5\ cm/s)(1 \times 10^{-6}\ s) = 0.16\ cm\ .$$

2 *Purcell 9.5   Consider the wave in free space*

$$E_x = 0 \qquad E_y = E_o \sin(kx - \omega t) \qquad E_z = 0$$

$$B_x = 0 \qquad B_y = 0 \qquad B_z = -E_o \sin(kx - \omega t)\ .$$

Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way. Suppose $\omega = 10^{10}\ hz$ and $E_o = 0.05\ statvolt/cm$. What is the wavelength in cm? What is the energy density in $ergs/cm^3$, averaged over a large region? From this calculate the power density, the energy flow in $ergs/cm^2 s$.

We see immediately that $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$. It's also easy to calculate

$$\nabla \times \mathbf{E} = kE_o \cos(kx + \omega t)\, \hat{\mathbf{z}} \quad,$$

$$\nabla \times \mathbf{B} = kE_o \cos(kx + \omega t)\, \hat{\mathbf{y}} \quad,$$

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \frac{\omega}{c}E_o \cos(kx + \omega t)\, \hat{\mathbf{z}} \quad,$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{\omega}{c}E_o \cos(kx - \omega t)\, \hat{\mathbf{y}} \quad.$$

The other two Maxwell's equations will be satisfied if

$$c = \frac{\omega}{k} \quad.$$

In this case,

$$k = \frac{2\pi}{\lambda} = \frac{10^{10}\ 1/s}{3 \times 10^{10}\ cm/s} = \frac{1}{3\ cm} \quad,$$

$$\lambda = 6\pi\ cm = 18.8\ cm \quad.$$

The average energy density is

$$\frac{E_o^2}{8\pi} = \frac{(0.05\ statvolt/cm)^2}{8\pi} = 9.95 \times 10^{-5}\ erg/cm^3 \quad,$$

and the average intensity (= power density) is

$$\frac{cE_o^2}{8\pi} = 3 \times 10^6\ erg/cm^2 s \quad.$$

3 *Purcell 9.9   The cosmic microwave background radiation apparently fills all space with an energy density of $4 \times 10^{-13}\ erg/cm^3$. Calculate the rms electric field strength in statvolt/cm and in V/m. Roughly how far away from a 1 kW radio transmitter would you find a comparable electromagnetic wave intensity?*

The average energy density is

$$\frac{E_{rms}^2}{4\pi} = 4 \times 10^{-13}\ erg/cm^3 \quad,$$

and

$$E_{rms} = 2.2 \times 10^{-6}\ statvolt/cm = 0.067\ V/m \quad.$$

Assuming the transmitter projects in all directions, a distance r away, the intensity of the transmitter is

$$\frac{1\ kW}{4\pi r^2} = \frac{E_{rms}^2}{\sqrt{\mu_o/\epsilon_o}} = \frac{(0.067\ V/m)^2}{376.73\ ohms} \quad,$$

$$r = 2584\ m \quad.$$

4 *Purcell 9.10*   *Find the magnetic field at a point P midway between the plates of capacitor a distance r from the axis of symmetry. A current I is flowing through the capacitor.*

We assume that the magnetic field circles the capacitor axis. If the capacitor spacing is small, the electric field will be fairly uniform and

$$E = \frac{V}{s} = \frac{Q}{sC} = \frac{4\pi s Q}{s\pi b^2} = \frac{4Q}{b^2} \quad ,$$

$$\Phi_E = \pi r^2 E = \frac{4\pi r^2 Q}{b^2} \quad ,$$

and ignoring signs,

$$2\pi r B = \frac{1}{c}\frac{4\pi r^2}{b^2}\frac{dQ}{dt} \quad ,$$

$$B = \frac{2rI}{cb^2} \quad .$$

At the edge of the capacitor $(r = b)$ this is the same as the magnetic field around a long wire.

5 *Purcell 10.7*   *A cell membrane typically has a capacitance around 1 $\mu F/cm^2$. It is believed the membrane consists of material having a dielectric constant of about 3. Find the thickness this implies. Other electrical measurements have indicated that the resistance of 1 cm^2 of cell membrane is around 1000 ohms. Show that the time constant of such a leaky capacitor is independent of the area of the capacitor. How large is it in this case? What is the resistivity?*

The capacitance is given in Farads so we will use SI. The constant ϵ_o appears in SI formulas. To deal with a dielectric material, we make the replacement $\epsilon_o \rightarrow \epsilon$. However, in SI, ϵ is not dimensionless. For example, if $\epsilon = 3$ in cgs, the value in SI is $3\epsilon_o$. The capacitance of a parallel plate capacitor is

$$C = \epsilon\frac{A}{s} = 3\frac{1\ cm^2}{s} = 1 \times 10^{-6}\ F \quad ,$$

so

$$s = 2.66 \times 10^{-9}\ m \quad .$$

We may view the leaky capacitor as a simple RC circuit, where the resistor and the capacitor are really the same element. The time constant is

$$\tau = RC = \frac{\rho s}{A}\epsilon\frac{A}{s} = \rho\epsilon \quad ,$$

which is independent of the area of the membrane. It is also independent of its thickness..

$$\tau = (1000\ ohms)(1 \times 10^{-6}\ F) = 1 \times 10^{-3}\ s = \rho\,3(8.85 \times 10^{-12}\ C^2/Nm^2)$$

$$\rho = 3.8 \times 10^7\ ohm\,m$$

$$\tau = sec$$

6 Purcell 10.17

In the first two cases, we assume the left dipole to be present and we bring in the right dipole from infinity. We would like to do this in such a way that the work required is zero. This will be the case if the path of the dipole coming in is perpendicular to the force on it. We will bring in the second dipole on a straight line from the right. The force on it is

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E} \ ,$$

where $\mathbf{E}$ is the field created by the other dipole. We wish to find an orientation for the right dipole so that this force is perpendicular to the path. We can do this if the dipole is pointed to the right. In this case the force is

$$\mathbf{F} = p\frac{\partial \mathbf{E}}{\partial x} \ .$$

The field from the left dipole on the line of the path is

$$\mathbf{E} = -\frac{p}{r^3}\hat{\mathbf{z}} = -\frac{p}{x^3}\hat{\mathbf{z}} \ .$$

The force is then

$$\mathbf{F} = 3\frac{p^2}{x^4}\hat{\mathbf{z}} \ ,$$

which is perpendicular to the path. Intuitively we can think of the dipole as two charges. The positive charge feels a force down and the negative charge feels a force up. But the positive charge is further away so the force on it is smaller. The net force is then up. But since this is perpendicular to the path, it still requires no work to bring in the dipole.

The field at the right dipole is

$$\mathbf{E} = -\frac{p}{d^3}\hat{\mathbf{z}} \ ,$$

and the torque exerted on this is

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} = pE \sin\theta\, \hat{\mathbf{y}} \ .$$

The work we do in rotating the dipole is minus the work done by the field. So

$$W = \int_{\pi/2}^{\pi} pE \sin\theta\, d\theta = pE = \frac{p^2}{d^3} \ .$$

This is good since the dipoles don't like to point in the same direction in this orientation so it should take positive work to arrange it.

In the second case, we rotate the dipole the same amount in the opposite direction and do the opposite work.

$$W = -pE = -\frac{p^2}{d^3}$$

In the third situation, the field from the left dipole points to the right along the path, so we can't bring in the second dipole pointing to the right. However, if we bring it in pointing up, the

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force on it will be up, perpendicular to the path. Taking the derivative is a little messy in this case, but we can find the direction of the derivative intuitively. If you think of the dipole as two charges, the force on the positive charge will be up to the right, and the force on the negative charge will be up to the left. But since the field is symmetric with respect to the x axis, the x components cancel out leaving a force up.

The field at the second dipole in this case has strength $E = 2p/d^3$. In analogy with the above results, the work to rotate it to the right is

$$W = -2\frac{p^2}{d^3} \quad,$$

because the dipole wants to be in this orientation. The work for the final situation is then

$$W = 2\frac{p^2}{d^3} \quad.$$

7 Purcell 10.19

If the ion is positive, the dipole will point away from it. The dipole field at the ion location will then point towards the dipole and the force will be attractive. If the ion is negative, the dipole will point towards it and the dipole field at the ion will be away from the dipole. The force again will be attractive. The polarization is $p = \alpha E = \alpha q/r^2$ where q is the ion charge. The force on the ion is

$$F = q\frac{2p}{r^3} = \frac{2q^2\alpha}{r^5} \quad.$$

To find the potential energy, we bring in the ion from infinity. The work required is

$$U = -\int_\infty^r \frac{2\alpha q^2}{r^5}dr = -\frac{\alpha q^2}{2r^4} \quad.$$

For sodium, $\alpha = 27 \times 10^{-24}$ cm^3.

$$4 \times 10^{-14} \; erg = \frac{(4.8 \times 10^{-10} \; esu)^2 (27 \times 10^{-24} \; cm^3)}{2r^4}$$

$$r = 9.4 \times 10^{-8} \; cm$$

8 *Purcell 10.21*

The maximum field strength is

$$E_m = \frac{14 \times 10^3\ V}{0.0000254\ m} = 5.5 \times 10^8\ V/m \quad .$$

The energy density is

$$\frac{1}{2}\epsilon E_m^2 = \frac{1}{2}(3.25)(8.85 \times 10^{-12}\ C^2/N\,m^2)(5.5 \times 10^8\ V/m)^2 = 4.35 \times 10^6\ J/m^3 \quad .$$

Remember that in SI, we need to insert the ϵ_o. The energy per mass is then

$$\frac{4.35 \times 10^6\ J/m^3}{1400\ kg/m^3} = 3100\ J/kg \quad .$$

This can raise the capacitor to a height

$$mgh = (0.75)m(3107\ J/kg) \quad ,$$

$$h = 238\ m \quad .$$